M.Math. Ist year
Second semestral exam 2010
Differential geometry I
B.Sury
Answer SIX questions including 1 and 2

Q 1.
Let α be a smooth space curve of unit speed. Assume that its curvature $K(t)$ is nowhere zero. Consider the curve $\beta(t) = \alpha'(t)$. If s is an arc length parameter for β, then prove that $\frac{ds}{dt} = K$. Further, show that the curvature of β is $(1 + \tau^2/K^2)^{1/2}$.

OR

Let α be a space curve with non-vanishing curvature parametrized by arc length. Suppose the principal normal vector $N(t) = f(t)\alpha(t)$ for all t where f is a smooth function. Prove that the curve must be part of a circle.

Q 2.
Consider the curve $\alpha(t) := (e^t \cos t, e^t \sin t, e^t); 0 \leq t \leq \pi$ contained in the local parametrization $f(u, v) = (u \cos v, u \sin v, u)$ of the cone. Prove that its length is $\sqrt{3}(e^\pi - 1)$.

OR

Let α be a plane curve with signed normal $N_{sign}(t)$ and signed curvature function $K_{sign}(t)$. For a constant λ, look at the 'parallel' plane curve α_λ; that is, $\alpha_\lambda(t) = \alpha(t) + \lambda N_{sign}(t)$. If $|\lambda K_{sign}(t)| < 1$ for all t, show that the curve α_λ is regular and its signed curvature is $\frac{K_{sign}}{1-\lambda K_{sign}}$.

Q 3.
(i) Show that the hyperboloid $S = \{(x, y, z) : x^2 + y^2 = 1 + z^2\}$ equals the union of the lines L_θ as θ varies in $[0, \pi)$, where L_θ is

$$(x - z)\cos\theta = (1 - y)\sin\theta$$

$$(x + z)\cos\theta = (1 + y)\sin\theta$$

(ii) Prove that a compact surface cannot be covered by a single parametrization.

Q 4.

(i) Prove that applying a translation or rotation of $\mathbf{R}^3$ to a local parametrization of a surface S does not change the first fundamental form.

(ii) If a local parametrization $f(u,v)$ of a surface is re-parametrized as $(u,v) \mapsto (u_1, v_1)$, the first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ changes to $E_1 du_1^2 + 2F_1 du_1 dv_1 + G dv_1^2$. Show that $\begin{pmatrix} E_1 & F_1 \\ F_1 & G_1 \end{pmatrix} = A^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} A$ for a matrix A.

Q 5.

Consider a tangent developable S of a curve α of unit speed; that is, a parametrization is $f(u,v) = \alpha(u) + v\alpha'(u)$. Show that the first fundamental form is $(1 + v^2K^2)du^2 + 2dudv + dv^2$.

OR

Consider the local parametrization

$$f(u,v) = (coshu \cos v, coshu \sin v, u), \quad 0 < v < 2\pi$$

of a part of a catenoid.

Consider a local parametrization

$$f_0(u,v) = (sinhu \cos(v), sinhu \sin(v), v)$$

of a part of a helicoid. Prove that $f(u,v) \mapsto f_0(u,v)$ is an isometry from a part of a catenoid to a part of a helicoid.

Hint: Show that the first fundamental forms are same.

Q 6.

Show that the area of the part $S = \{(x, y, x^2 + y^2) : x^2 + y^2 \leq 1\}$ of a paraboloid is $\frac{\pi}{6}(5^{3/2} - 1)$.

OR

Let $\alpha(t) = f(u(t), v(t))$ be a curve of not-necessarily-unit speed on $Im(f) \subset S$. Show that the normal curvature of α at any point $\alpha(t)$ is

$$K_n = \frac{Lu'^2 + 2Mu'v' + Nv'^2}{Eu'^2 + 2Fu'v' + Gv'^2}.$$

Q 7.
Let S be the torus covered by local parametrizations :

$$f(\theta, \phi) = ((a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta)$$

for $(\theta, \phi) \in I_1 \cup I_2 \cup I_3 \cup I_4$ where $I_1 = (0, 2\pi) \times (0, 2\pi)$, $I_2 = (0, 2\pi) \times (-\pi, \pi)$, $I_3 = (-\pi, \pi) \times (0, 2\pi)$, $I_4 = (-\pi, \pi) \times (-\pi, \pi)$.
(i) Choose any one of these patches and prove that the Gaussian curvature is

$$K(\theta, \phi) = \frac{\cos\theta}{b(a + b\cos\theta)}.$$

(ii) Assuming (i) for all patches, show that $\int_0^{2\pi} \int_0^{2\pi} K\sqrt{FG - F^2}\, d\theta d\phi = 0$.

Q 8.
Let α be a parametrized curve of unit speed contained in a local parametrization $f : U \to S$ of a surface S. Recall that α is said to be a geodesic if the acceleration α'' is in the direction of the standard unit normal corresponding to f. Writing $\alpha(t) = f(u(t), v(t))$, obtain equations in terms of u, v which characterize geodesics.

OR

Show that the mean curvature H of the surface $f(u, v) = (u, v, \log(\frac{\cos v}{\cos u}))$ is zero. You may use the expression $H = \frac{LG - 2MF + NE}{2(EG - F^2)}$ if necessary.